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Ramsey Properties of Random Hypergraphs

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Let $K^{(k)}(n, p)$ be the random k-uniform hypergraph obtained by independent inclusion of each of the $\binom{n}{k}$ k-tuples with probability p. For an arbitrary k-uniform

$k=3$, $r=2$, and $G=K_4^3$. In the proof we utilize a recent version of the hypergraph regularity lemma due to Frankl and Rödl. © 1998 Academic Press

1. INTRODUCTION

Let F and G be two k-uniform hypergraphs. The arrow notation $F \to (G)_r^e$ ($F \to (G)_r^v$) frequently used in Ramsey theory abbreviates the following fact: *For every partition of the edges* (*vertices*) *of F into r classes, at least one of the classes contains a copy of G.*

Although the classical Ramsey results do not involve explicitly random structures, probabilistic methods have been successfully used for a long time (e.g. [4]). On the other hand, Ramsey properties of random structures have been studied only recently. The problem of finding thresholds for Ramsey properties of the binomial random graph $K(n, p)$ was settled in [9, 10, 12].

For a graph G with at least three vertices, define the parameters

$$m_G^1 = \max_{H \subseteq G,\, v_H \geq 2} \frac{e_H}{v_H - 1} \qquad \text{and} \qquad m_G^2 = \max_{H \subseteq G,\, v_H \geq 3} \frac{e_H - 1}{v_H - 2},$$

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where v_H and e_H stand for the number of vertices and edges of a graph H, respectively. The following two theorems characterize Ramsey properties of the random graph $K(n, p)$.

THEOREM 1.1 [9]. *For every integer r, $r \geqslant 2$, and for every graph G which, in case $r = 2$, is not a matching, there exist constants c and C such that*

$$\lim_{n \to \infty} \mathrm{P}(K(n, p) \to (G)_r^v) = \begin{cases} 1 & \text{if} \quad p > Cn^{-1/m_G^1} \\ 0 & \text{if} \quad p < cn^{-1/m_G^1}. \end{cases}$$

THEOREM 1.2 [10, 12]. *For every integer r, $r \geqslant 2$, and for every graph G which is not a star forest, there exist constants c and C such that*

$$\lim_{n \to \infty} \mathrm{P}(K(n, p) \to (G)_r^e) = \begin{cases} 1 & \text{if} \quad p > Cn^{-1/m_G^2} \\ 0 & \text{if} \quad p < cn^{-1/m_G^2}. \end{cases}$$

The aim of this paper is to continue this research and investigate the Ramsey properties of random hypergraphs. A random k-uniform hypergraph $K^{(k)}(n, p)$ is one where each out of $\binom{n}{k}$ k-tuples is included as an edge independently with probability p. In Section 3 we will extend Theorem 1.1 to k-uniform hypergraphs using basically the same approach as in [9]. The similar extension of Theorem 1.2 seems to be far from obvious and we give a partial solution only.

The most challenging problem here is to prove the following positive statement:

Conjecture 1.3. For every k-uniform hypergraph G and integers $r \geqslant 2$ and $k \geqslant 3$ there exists $C > 0$ such that

$$\lim_{n \to \infty} \mathrm{P}(K^{(k)}(n, p) \to (G)_r^e) = 1$$

if $p \geqslant p_0$, where $p_0 = Cn^{-1/m_G^k}$ and $m_G^k = \max_{H \subseteq G} ((e_H - 1)/(v_H - k))$.

A heuristic reason behind this conjecture is that for $p \geqslant p_0$ the edges of $K^{(k)}(n, p)$ are, on average, contained in many (read, large constant) copies of G, which, we believe, is a necessary and sufficient condition for $K^{(k)}(n, p) \to (G)_r^e$.

Note. The negative counterpart statement to Conjecture 1.3, saying that there exists $c > 0$ such that

$$\lim_{n \to \infty} \mathrm{P}(K^{(k)}(n, p) \to (G)_r^e) = 0$$

if $p \leqslant p_0$, where $p_0 = cn^{-1/m_G^k}$, is not considered here. We believe, however, that its proof would follow the lines of the argument from [10].

In this paper we confirm Conjecture 1.3 in the first nontrivial case, when $G = K_4^{(3)}$, a complete 3-uniform hypergraph on four vertices, and $r = 2$. Namely, we prove the following theorem.

THEOREM 1.4. *There exists an absolute constant* $C > 0$ *such that*

$$\lim_{n \to \infty} \mathrm{P}(K^{(3)}(n, p) \to (K_4^{(3)})_2^{(e)}) = 1$$

for $p > Cn^{-1/3}$.

In the case of graphs, the essential tool in the proof was the Szemerédi Regularity Lemma. Our approach here, based in part on ideas used in [12], utilizes some results about the regularity of 3-uniform hypergraphs proved recently in [6].

Our paper is organized as follows. The next section contains preliminary results on exponentially small probabilities, including Janson's inequality. In Sections 3 and 5, respectively, we present our proofs of the extension of Theorem 1.1 to random k-uniform hypergraphs (Theorem 3.1) and of Theorem 1.4, which we consider as our main result here. In Section 4 we provide regularity lemmas for both graphs and hypergraphs, which are so crucial for our argument. Finally, the Appendix contains a proof of a strengthening of our previous result from [11] about monochromatic triangles in random graphs. This stronger form is needed for the proof of Theorem 1.4.

Throughout the paper we adopt the standard Ramsey theory notation $[A]^t$ for the family of all t-elements subsets of a given set A.

2. EXPONENTIALLY SMALL PROBABILITIES

Let X be a random variable with the binomial distribution with expectation np. Chebyshev's inequality asserts that for every $\varepsilon > 0$

$$P(|X - np| > \varepsilon np) < \frac{1}{\varepsilon^2 np}.$$

A much better bound is provided by the Chernoff inequality

$$P(|X - np| > \varepsilon np) < \exp\{-\mathrm{Ch}(\varepsilon)\, np\},$$

where $\mathrm{Ch}(\varepsilon)$ is a positive constant.

In many places in our proofs we need exponentially small bounds on tails of sums of not necessarily independent random variables. If the dependence is relatively weak, the bounds for the lower tail are provided by Janson's inequality, which is an extension of an inequality from [8]. We shall formulate them both in a quite general form. Throughout the paper $\mathrm{E}(X)$ stands for the expectation of a random variable X and should not be confused with the plain notation $E(G)$ for the edge set of a graph G. Let F be a finite set, from which a subset is drawn randomly in such a way that the inclusions of individual elements are independent. Further, let $\mathscr{S}$ be a family of subsets of F and for each $A \in \mathscr{S}$ let I_A equal 1 if A is entirely included in the random subset and 0 otherwise. Finally, let $X = \sum_{A \in \mathscr{S}} I_A$. Then

Lemma 2.1 [8].

$$\mathrm{P}(X=0) \leqslant \exp\left\{-\frac{\mathrm{E}(X)^2}{\sum\sum_{A \cap B \neq \varnothing} \mathrm{E}(I_A I_B)}\right\}.$$

Janson generalized this inequality to a lower tail bound:

Lemma 2.2 [7]. *For every* $0 < \varepsilon \leqslant 1$,

$$\mathrm{P}(X \leqslant (1-\varepsilon)\,\mathrm{E}(X)) \leqslant \exp\left\{-\frac{(\varepsilon \mathrm{E}(X))^2}{2\sum\sum_{A \cap B \neq \varnothing} \mathrm{E}(I_A I_B)}\right\}.$$

Unfortunately, the upper tail counterpart of Lemma 2.2 is not true in general. As an exponential bound is often needed also for the upper tail, to cope with this situation, we developed in [12] an approach based on the following elementary lemma, which deals with a somewhat simplified case when all elements are included in the random set with the same probability p and all members of $\mathscr{S}$ are of the same size s.

Lemma 2.3 [12]. *Let F be a finite set and $\mathscr{S}$ a family of s-element subsets of F. For $0 < p < 1$, let F_p be a random subset of F obtained by independent inclusion of each element with probability p. Then, for any integer k, with probability at least $1 - 2^{-k/s}$, there exists a set $E \subset F_p$ of size k such that $F_p \backslash E$ contains at most $2\,|\mathscr{S}|\,p^s$ sets from $\mathscr{S}$.*

Hence, exceeding two times the expectation is exponentially unlikely, provided we are allowed to destroy some of the objects in count, by deleting a certain number of elements from the random set. Then, of course, there is a danger of losing other properties held by the random set. It turns out, however, that monotone properties, held with exponential probabilities, survive the deletion. The next lemma, also from [12], makes it precise.

For a family $\mathcal{Q}$ of subsets of a set F and an integer k, let

$$\mathcal{Q}_k = \{A \colon \forall B \subseteq A, \text{ if } |B| \leqslant k, \text{ then } A \backslash B \in \mathcal{Q}\}.$$

LEMMA 2.4 [12]. *Let F be a set of m elements, $0 < p < 1$, and b and δ satisfy*

$$\delta(1 + \log_2 e - \log_2 \delta) < (1 - \delta) b. \tag{2.1}$$

Then, for every increasing family $\mathcal{Q} = \mathcal{Q}(m)$ of subsets of F and for $0 < k \leqslant \delta mp/2$, if $\mathrm{P}(F_{(1-\delta)p} \in \neg \mathcal{Q}) < 2^{-(1-\delta) bmp}$ then $\mathrm{P}(F_p \in \neg \mathcal{Q}_k) < 2^{-b'mp}$, provided mp is large enough, where $b' = b'(b, \delta) = \frac{1}{2}\min\{(1-\delta) b/2, (\log_2 e) \operatorname{Ch}(\delta/2)\}$.

Warning. The property $\mathcal{Q}$ must not depend on p, but on m only.

These two lemmas complement each other and for future references we derive a corollary from them.

COROLLARY 2.5. *Let F be a set, $|F| = m$, $0 < p < 1$, and let δ and b satisfy inequality (2.1). Furthermore, let $p_0 = (1 - \delta) p$, where $p = p(m)$ and $pm \to \infty$, and let $k = \frac{1}{2}\delta mp$ be an integer. Let $\mathcal{S}$ be a family of s-element subsets of F and $\mathcal{Q} = \mathcal{Q}(m)$ be an increasing family of subsets of F. Then, for m large enough and with b' as in Lemma 2.4, if*

$$\mathrm{P}(F_{p_0} \in \neg \mathcal{Q}) < 2^{-bmp_0}$$

then, with probability at least

$$1 - 2^{-k/s} - 2^{-b'mp},$$

there exists a set $E_0 \subset F_p$, $|E_0| = k$, such that

(i) *$F_p \backslash E_0 \in \mathcal{Q}$, and*

(ii) *$F_p \backslash E_0$ contains at most $2\,|\mathcal{S}|\, p^s$ sets from $\mathcal{S}$.*

Proof. As $\mathrm{P}(F_{p_0} \in \neg \mathcal{Q}) < 2^{-bmp_0}$, we infer by Lemma 2.4 that $\mathrm{P}(F_p \in \neg \mathcal{Q}_k) < 2^{-b'mp}$. In other words, with probability at least $1 - 2^{-b'mp}$, we have that $F_p \backslash E \in \mathcal{Q}$ for all $|E| \leqslant k$.

On the other hand, by Lemma 2.3, with probability at least $1 - 2^{-k/s}$, there is a set E_0, $|E_0| \leqslant k$, such that $F_p \backslash E_0$ contains at most $2\,|\mathcal{S}|\, p^s$ sets from $\mathcal{S}$.

Combining these two facts, we obtain that with probability at least $1 - 2^{-k/s} - 2^{-b'mp}$ there exists a set $E_0 \subset F_p$, $|E_0| = k$, such that (i) and (ii) hold. ∎

Occasionally, when $\mathrm{P}(F_p \in \mathcal{Q}(m)) \to 1$ as $m \to \infty$, we will be using the phrase "F_p possesses the property $\mathcal{Q}$ *almost surely*".

3. VERTEX COLORING

The aim of this section is to prove an extension of Theorem 1.1 to k-uniform hypergraphs. For a k-uniform hypergraph G, let $m_G^1 = \max_{H \subseteq G}(e_H/(v_H - 1))$.

THEOREM 3.1. *For every integer r, $r \geqslant 2$, and k, $k \geqslant 3$, and for every hypergraph G, there exist constants c and C such that*

$$\lim_{n \to \infty} \mathrm{P}(K^{(k)}(n, p) \to (G)_r^v) = \begin{cases} 1 & \text{if } \ p > Cn^{-1/m_G^1} \\ 0 & \text{if } \ p < cn^{-1/m_G^1}. \end{cases}$$

Let G be a k-uniform hypergraph. Denote by X_G the random variable counting the copies of G in $K^{(k)}(n, p)$ and let

$$\Phi_G = \min_{H \subseteq G,\, e_H > 0} \mathrm{E}(X_H).$$

Note that

$$\mathrm{E}(X_H) = \Theta(n^{v_H} p^{e_H})$$

and that, denoting by $I_{G'}$ the indicator that G', a copy of G in $[n]^k$, belongs to $\mathrm{P}(K^{(k)}(n, p))$, we have

$$\sum \sum_{G' \cap G'' \neq \varnothing} \mathrm{E}(I_{G'} I_{G''}) = O\left(\sum_{H \subseteq G,\, e_H > 0} n^{2v_G - v_H} p^{2e_G - e_H} \right).$$

Thus, in this special case, Lemma 2.1 says that there exists a constant c_G such that

$$\mathrm{P}(K^{(k)}(n, p) \not\supset G) = \mathrm{P}(X_G = 0) < \exp\{-c_G \Phi_G\}.$$

Equipped with this tool we now give a short proof of the positive statement of Theorem 3.1 followed by the proof of its negative part.

Proof of Theorem 3.1. Suppose that $K^{(k)}(n, p) \nrightarrow (G)_r^v$. Then the largest color class of any coloring with no monochromatic G spans a G-free subhypergraph of size at least n/r. The probability that this happens is, using the above consequence of Lemma 2.1, smaller than

$$2^n \mathrm{P}(K^{(k)}(n/r, p) \not\supset G) < 2^n e^{-c_G \Phi_G},$$

where Φ_G is with respect to $K^{(k)}(n/r, p)$ rather than $K^{(k)}(n, p)$.

For each $H \subseteq G$, $e_H > 0$, we have

$$\mathrm{E}(X_H) = \Theta(n^{v_H} p^{e_H}) = \Theta(n(np^{e_H/(v_H-1)})^{v_H-1}) > \Theta(C^{m_G^1(v_H-1)} n),$$

so that the probability of $K^{(k)}(n, p) \nrightarrow (G)_r^v$ tends to 0 for C sufficiently large.

For the proof of the negative part of Theorem 3.1 we assume that $p < cn^{-1/m_G^1}$, where c is a sufficiently small constant. Throughout we shall be referring to subhypergraphs as subgraphs.

As for every three hypergraphs F, G, and H, where $H \subset G$, and $F \rightarrow (G)_r^v$ implies $F \rightarrow (H)_r^v$, without loss of generality we may assume that for every proper subgraph H of G with at least 2 vertices

$$\frac{e_H}{v_H - 1} < \frac{e_G}{v_G - 1}.$$

(If this was not the case, one could replace G with its smallest subgraph H satisfying $e_H/(v_H - 1) = m_G^1$.) This assumption implies that there are no isolated vertices in G and also that for each proper subgraph H of G with at least 2 vertices,

$$n^{v_H - 1} p^{e_H} = \Omega(n^{\varepsilon}) \tag{3.1}$$

for some $\varepsilon > 0$.

Our proof will consist of two statements, one deterministic, saying that the property $F \rightarrow (G)_2^v$ implies the existence of a certain structure in F, while the probabilistic statement will almost surely exclude that structure from the random hypergraph $K^{(k)}(n, p)$. We shall need a few definitions first.

A simple path is a hypergraph consisting of edges $E_1, ..., E_l$, $l \geqslant 1$, such that

$$|E_i \cap E_j| = \begin{cases} 1 & \text{if } \; j = i+1, \quad i = 1, ..., l-1 \\ 0 & \text{otherwise.} \end{cases}$$

A fairly simple (*simple*) *cycle* is a hypergraph which consists of a simple path $(E_1, ..., E_l)$, $l \geqslant 2$, and an edge E_0 such that

$$|E_0 \cap E_i| = \begin{cases} 1 & \text{if } \; i = 1 \\ 0 & \text{for } \; i = 2, ..., l-1 \\ s & \text{if } \; i = l, \end{cases}$$

where $s \geqslant 1$ ($s = 1$, respectively). A fairly simple but not simple cycle will be called *spoiled*.

For hypergraphs F and G, let $H(F, G)$ be the hypergraph with the vertex set $V(F)$ whose edges are the vertex sets of all copies of G which are contained in F.

We shall call this hypergraph *a superhypergraph* and its edges *superedges* in order to distinguish them from the edges of F or G. We shall be saying that an edge belongs to a superedge if the copy of G whose vertex set makes that superedge contains the said edge.

A subsuperhypergraph H_0 of $H(F, G)$ is said to have *a handle* if there is a superedge E in $H(F, G)$ such that $|E \cap V(H_0)| \geqslant 2$ and, at least one edge which belongs to E does not belong to any superedge of H_0.

Deterministic Lemma. *If $F \to (G)_2^v$ and G has no isolated vertex then the superhypergraph $H(F, G)$ contains a fairly simple cycle with a handle.*

Probabilistic Lemma. *If p and G are as above then, almost surely, the random super-hypergraph $H(K^{(k)}(n, p), G)$ contains no fairly simple cycle with a handle.*

The Proof of the Deterministic Lemma. Assume that $F \to (G)_2^v$. This is equivalent to saying that the chromatic number of $H(F, G)$ is at least 3. We may assume that $H(F, G)$ is edge-critical with respect to that property or otherwise we could replace $H(F, G)$ with its 3-edge-critical subgraph, ignoring some copies of G in F. As such, it satisfies a certain property which we now formulate as an exercise for the reader.

Exercise. Show that if H is a 3-edge-critical hypergraph then for every edge $E \in H$ and for every vertex $v \in E$ there is $E' \in H$ such that $E \cap E' = \{x\}$. (*Hint*: By 3-edge-criticality there exists a coloring with only the edge E monochromatic. Try to switch the color of x.)

Let P be the longest simple path in $H = H(F, G)$. By the Exercise, P contains at least two superedges of H. Let x and y be two vertices which belong to only the first superedge of P, and let E_x and E_y be two superedges of H (read: copies of G) whose existence is guaranteed by Exercise, i.e. $E_z \cap E_1 = \{z\}$, $z = x, y$.

By the maximality of P, $|V(P) \cap E_z| \geqslant 2$, $z = x, y$. Let $i_z = \min\{i \geqslant 2: E_z \cap E_i \neq \varnothing\}$, $z = x, y$, and assume that, say, $i_y \leqslant i_x$. The superedges $E_1, ..., E_{i_x}, E_x$ form a fairly simple cycle for which E_y is a handle, as no edge of E_y containing y can belong to any superedge of P or to E_x. ∎

The Proof of the Probabilistic Lemma. Let X, Y, and Z be random variables counting, respectively, simple paths of length at least $B \log n$, spoiled cycles, and simple cycles of length less than $B \log n + 1$ with handles, in the random superhypergraph $H(K^{(k)}(n, p), G)$, where $B = B(c, G)$ is a big enough constant. Straightforward estimates show that their expectations all converge to 0 as $n \to \infty$. Indeed,

$$\mathrm{E}(X) < \sum_{t > B \log n} n^{t(v_G - 1) + 1} p^{te_G} < n \sum_{t > B \log n} (c^{e_G})^t = o(1),$$

$$\mathrm{E}(Y) < \sum_{t > 2} \sum_{H \subset G} n^{t(v_G - 1) - (v_H - 1)} p^{te_G - e_H} = o(1),$$

and

$$\mathrm{E}(Z) = O\left(\sum_{t=3}^{B \log n} \sum_{H \subset G} n^{(t+1)(v_G - 1) - (v_H - 1)} (\log n)^{v_H} p^{(t+1) e_G - e_H} \right) = o(1),$$

where the inner sum extends over all proper subgraphs H of G with at least 2 vertices and corresponds, in case of Y, to all possible shapes of the intersection of the last edge of a cycle with the previous edge, and, in case of Z, to all possible shapes of the intersection of the handle with the cycle. The index t stands for the number of superedges in a path or cycle. The logarithmic factor in the last estimate represents the number of choices of the vertices at which a handle is attached to the cycle. Finally, we made use of formula (3.1) here.

Thus, by Markov's inequality, $\mathrm{P}(X = Y = Z = 0) \to 1$ as $n \to \infty$, which was to be proved. ∎

Comment. In case $k = 2$, the proof of the negative part of Theorem 3.1 contained in [9] relied on a deterministic lemma, true in fact for any $k \geqslant 2$, which can be formulated as follows:

$$\text{If} \quad m_F < \frac{2}{k} \max_{H \subseteq G} \delta_H \qquad \text{then} \quad F \nrightarrow (G)_2^v.$$

However, in order to extend that original proof to the case $k \geqslant 3$, one needs another, though similar result.

PROPOSITION 3.2. *If $m_F \leqslant m_G^1$ then $F \nrightarrow (G)_2^v$.*

While it is possible to prove Proposition 3.2 using a deterministic version of the approach applied in the proof above, we shall outline here a probabilistic proof deriving Proposition 3.2 from Theorem 3.1

A Probabilistic Proof of Proposition 3.2 (*Outline*). Let F satisfy the inequality $m_F \leqslant m_G^1$. Consider a random hypergraph $K^{(k)}(n, p)$ with $p = cn^{-1/m_G^1}$, where c is so small that the negative part of Theorem 3.1

applies. Using the standard second moment method one can show that, for some $\alpha > 0$,

$$\limsup_{n \to \infty} \mathrm{P}(K^{(k)}(n, p) \supset F) > \alpha.$$

As, on the other hand, by Theorem 3.1,

$$\lim_{n \to \infty} \mathrm{P}(K^{(k)}(n, p) \not\to (G)_2^v) = 1,$$

there exists an n-vertex graph Γ such that $\Gamma \supset F$ and $\Gamma \not\to (G)_2^v$. But then also $F \not\to (G)_2^v$. ∎

4. REGULARITY OF GRAPHS AND HYPERGRAPHS

In this section we collected results about ε-regular graphs, including the celebrated Szemerédi Regularity Lemma, and analogous notions and statements for hypergraphs, among them a recent hypergraph regularity lemma due to Frankl and Rödl.

For two disjoint subsets A and B of the vertex set of a graph, let $e(A, B)$ denote the number of edges between them and let $d(A, B) = (e(A, B))/(|A|\,|B|)$.

We say that a bipartite graph $G = (X, Y, E)$ is $(\varepsilon_1, \varepsilon_2, d)$-*regular* if for all $A \subseteq X$ and $B \subseteq Y$ with $|A| > \varepsilon_1 |X|$ and $|B| > \varepsilon_1 |Y|$, we have $|d(A, B) - d| < \varepsilon_2$. If $\varepsilon_1 = \varepsilon_2 = \varepsilon$, we use the name (ε, d)-*regular*. If furthermore, d is not specified, one always assumes that $d = d(X, Y)$, and calls such graphs ε -*regular*.

A partition $V = V_1 \cup V_2 \cup \cdots \cup V_t$ is called *equable* if the sizes of the sets V_i, $i = 1, ..., t$, differ from each other by at most 1.

A partition of the vertex set of a graph into t parts is called ε-*regular* if all but at most $\varepsilon\binom{t}{2}$ pairs of partition sets span ε-regular subgraphs.

The following version of Szemerédi's regularity lemma ([14]) will be utilized in the Appendix.

THEOREM 4.1 (Szemerédi's Regularity Lemma). *For all $\varepsilon_0 > 0$, $t \geqslant 1$ and $r \geqslant 1$ there exist N and S such that for every family of r graphs on the same set V of at least N vertices and for every equable partition of V into t parts, $V = V_1 \cup V_2 \cup \cdots \cup V_t$, there exists an integer $s \leqslant S$ and equable partition of V into ts parts which refines the original partition and is ε_0-regular with respect to all r graphs.*

In the Appendix, we shall be applying this lemma with $r = 2$ and $t = 17$. With these two parameters fixed, the Szemerédi constant $S = S(\varepsilon_0)$ depends on ε_0 only.

This version differs from the original Szemerédi's theorem in two ways. It deals with r graphs rather than one and it also eliminates the exceptional class V_0.

First note that Szemerédi's proof allows an immediate extension to r graphs, by considering the index of a partition with respect to all r graphs.

Now we show how one can eliminate the exceptional class. We apply the original Szemerédi's regularity lemma simultaneously to r graphs and with ε_0^2, say, obtaining a partition of each V_i into $U_i^0, ... U_i^{s_i}$ such that all the sets U_i^j, $j \geqslant 1$, have the same size x and form an (ε_0^2, d)-regular partition with respect to all the graphs, whereas the sets U_i^0 are each smaller than $\varepsilon_0^2 n$, $n = |V_i|$, $i = 1, ..., t$. Setting $s = \lceil (n(1-\varepsilon_0^2))/x \rceil$, for each $i = 1, ..., t$, we redistribute the elements of $U_i^{s+1}, ..., U_i^{s_i}, U_i^0$ as equally as possible, among the sets $U_i^1, ..., U_i^s$. As observed already by Szemerédi in his original proof, the density $d(A, B)$ behaves in a continuous way, and adding such small bits does not affect the (ε_0, d)-regularity of the pairs (U_i^k, U_j^l) for $1 \leqslant k, l \leqslant s$.

The Szemerédi Regularity Lemma serves often to force the existence of many complete subgraphs of a given size. Here we shall concentrate on triangles only.

Let V_1, V_2 and V_3 be 3 disjoint subsets of V, $|V_1| = |V_2| = |V_3|$. Let P^{12}, P^{13} and P^{23} be bipartite graphs with bipartitions (V_1, V_2), (V_1, V_3) and (V_2, V_3), respectively. The triple $P = (P^{12}, P^{13}, P^{23})$ will be referred to as *a triad*. For a triad P, let

$$T(P) = \{xyz \colon x \in V_1, y \in V_2, z \in V_3, xy \in P^{12}, xz \in P^{13}, yz \in P^{23}\}$$

be the set of all triangles formed by the edges of $P^{12} \cup P^{13} \cup P^{23}$ and let $t(P) = |T(P)|$.

The following result is elementary.

Proposition 4.2. *Let $P = (P^{12}, P^{13}, P^{23})$ be a triad of ε-regular graphs on vertex sets V_1, V_2, V_3, of possibly different size, with densities ρ_{12}, ρ_{13} and ρ_{23}. Then*

$$(1-2\varepsilon)(\rho_{12}-\varepsilon)(\rho_{13}-\varepsilon)(\rho_{23}-\varepsilon) < \frac{t(P)}{|V_1|\,|V_2|\,|V_3|} < [2\varepsilon + (\rho_{12}+\varepsilon)(\rho_{13}+\varepsilon)(\rho_{23}+\varepsilon)].$$

In the proof of Theorem 1.4 we will make use of a regularity lemma for 3-uniform hypergraphs considered in [6]. First we introduce some concepts necessary for its formulation.

We will be interested in the following partial partition of $[V]^2$, where V is an arbitrary finite set.

DEFINITION 4.3. Let V be a finite set, l and t two positive integers, and ε_1 and ε_2 two positive real numbers. An $(l, t, \varepsilon_1, \varepsilon_2)$-*partition* $\mathscr{P}$ of $[V]^2$ consists of

(I) an auxiliary partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$, where $|V_0| < t$ and $|V_1| = |V_2| = \cdots = |V_t| \stackrel{\text{def}}{=} m$, together with

(II) a system of edge-disjoint bipartite graphs P^{ij}_α, $1 \leqslant i < j \leqslant t$, $0 \leqslant \alpha \leqslant l_{ij} \leqslant l$, with bipartitions (V_i, V_j), such that

(a) $|\bigcup_{\alpha=0}^{l_{ij}} P^{ij}_\alpha| = |V_i|\,|V_j|$ for all i, j, $1 \leqslant i < j \leqslant t$, and

(b) all but $\varepsilon_1\binom{t}{2} m^2$ pairs $\{v_i, v_j\}$, $v_i \in V_i$, $v_j \in V_j$, $1 \leqslant i < j \leqslant t$, are edges of ε_2-regular bipartite graphs P^{ij}_α, i.e. the total number of the edges of those P^{ij}_α's which are ε_2-regular is at least $(1-\varepsilon_1)\binom{t}{2} m^2$.

If, moreover,

(c) for all but $\varepsilon_1\binom{t}{2}$ pairs i, j, $1 \leqslant i < j \leqslant t$, $|P^{ij}_0| \leqslant \varepsilon_1 m^2$, and for all $\alpha \geqslant 1$, $(1/l) - \varepsilon_2 \leqslant d_{P^{ij}_\alpha}(V_i, V_j) \leqslant (1/l) + \varepsilon_2$ holds,

then we call such an $(l, t, \varepsilon_1, \varepsilon_2)$-partition *equitable*.

Equitable $(l, t, \varepsilon_1, \varepsilon_2)$-partitions play an analogous role to the vertex set partitions in the Szemerédi's Regularity Lemma for graphs. Now we impose some conditions that will describe regularity of 3-uniform hypergraphs.

Suppose that $H \subseteq [V]^3$ is a 3-uniform hypergraph and V_1, V_2, and V_3 are three disjoint subsets of V, $|V_1| = |V_2| = |V_3|$. Let P^{12}, P^{13}, and P^{23} be a triad of bipartite graphs with bipartitions (V_1, V_2), (V_1, V_3), and (V_2, V_3), respectively.

The density of H with respect to the triad $P = (P^{12}, P^{13}, P^{23})$ is defined by

$$d_H = \frac{|H \cap T(P)|}{t(P)},$$

and with respect to an r-tuple of triads $\vec{Q} = (Q_1, Q_2, ..., Q_r)$, by

$$d_H(\vec{Q}) = \left\{ \frac{|H \cap \bigcup_{s=1}^r T(Q_s)|}{|\bigcup_{s=1}^r T(Q_s)|} \right\},$$

where, recall, $T(P)$ is the number of triangles of P and $t(P) = |T(P)|$.

DEFINITION 4.4. Let r be an integer and let $\delta > 0$. We will say that a triad $P = (P^{12}, P^{13}, P^{23})$ is (δ, r)-*regular* with respect to a 3-uniform hypergraph H if for every r-tuple of triads $\vec{Q} = (Q_1, Q_2, ..., Q_r)$, $Q_s = (Q^{12}_s, Q^{13}_s, Q^{23}_s)$, $s = 1, 2, ...r$, where

$$Q^{12}_s \subseteq P^{12}, \qquad Q^{13}_s \subseteq P^{13}, \qquad Q^{23}_s \subseteq P^{23}, \qquad s = 1, 2, ..., r,$$

the following holds:

If $|\bigcup_{s=1}^r T(Q_s)| > \delta\,|T(P)|$ then $|d_H((\vec{Q}) - d_H(P)| < \delta$.

DEFINITION 4.5. For a 3-uniform hypergraph $H=(V,E)$, $|V|=n$, and an $(l, t, \varepsilon_1, \varepsilon_2)$-partition $\mathscr{P}$ of $[V]^2$, let $\mathscr{I}$ be the set of all (δ, r)-irregular traids formed by the bipartite graphs P^{ij}_α of the partition $\mathscr{P}$. We say that $\mathscr{P}$ is (δ, r)-*regular* with respect to H if

$$\sum_{P\in\mathscr{I}} t(P)<\delta n^3,$$

i.e. the number of triangles contained in the (δ, r)-irregular triads is a small (only 6δ) fraction of all $\binom{n}{3}$ triples.

Our next Proposition 4.6 asserts that there are not too many irregular graphs or triads in a (δ, r)-regular, equitable, $(l, t, \varepsilon_1, \varepsilon_2)$-partition.

PROPOSITION 4.6. *Let* $\mathscr{P}=\{P^{i,j}_\alpha,\ 0\leqslant\alpha\leqslant l_{ij}\leqslant l,\ 1\leqslant i<j\leqslant t\}$ *be a* (δ, r)-*regular, equitable,* $(l, t, \varepsilon_1, \varepsilon_2)$-*partition with respect to a hypergraph* H. *Let* $\mathscr{I}$ *be the set of all* (δ, r)-*irregular triads of* $\mathscr{P}$ *with respect to* H. *Then*

(i) *if* $\varepsilon_2<1/2l$ *then*

$$\sum_{1\leqslant i,j\leqslant t} |\{\alpha\geqslant 1\colon P^{i,j}_\alpha \textit{ is } (\varepsilon_2, 1/1)-\textit{irregular}\}|<3\varepsilon_1\binom{t}{2}l$$

and

(ii) *if* $(1-2\varepsilon_2)(1-2\varepsilon_2 l)^3>\frac{1}{2}$ *then* $|\mathscr{I}|<(2\delta+\varepsilon_1)\,t^3l^3$.

Proof. (i) By part (b) of Definition 4.3 we have that

$$\sum_{ij}\sum_{\alpha}\{|P^{ij}_\alpha|\colon 0\leqslant\alpha\leqslant l_{ij} \text{ and } P^{ij}_\alpha \text{ is } \varepsilon_2\text{-irregular}\}<\varepsilon_1\binom{t}{2}m^2. \tag{4.1}$$

Let $X\subseteq[t]^2$ be the set of all pairs $\{i,j\}$ for which the inequality

$$\frac{1}{l}-\varepsilon_2\leqslant d_{P^{ij}_\alpha}(V_i, V_j)\leqslant\frac{1}{l}+\varepsilon_2$$

from part (c) of Definition 4.3 holds for $\alpha\geqslant 1$. Then we have

$$\sum_{ij}\sum_{\alpha}\{|P^{ij}_\alpha|\colon 1\leqslant\alpha\leqslant l_{ij} \text{ and } P^{ij}_\alpha \text{ is } \varepsilon_2\text{-irregular}\}$$

$$\geqslant\sum_{ij\in X}|\{\alpha\geqslant 1\colon P^{ij}_\alpha \text{ is } \varepsilon_2\text{-irregular}\}|\left(\frac{1}{l}-\varepsilon_2\right)m^2. \tag{4.2}$$

Comparing (4.1) and (4.2), we infer that

$$\sum_{ij \in X} |\{\alpha \geqslant 1 : P_\alpha^{ij} \text{ is } \varepsilon_2\text{-irregular}\}| \leqslant \frac{\varepsilon_1 \binom{t}{2} l}{1-\varepsilon_2 l} < 2\varepsilon_1 \binom{t}{2} l. \tag{4.3}$$

On the other hand, due to the fact that $l_{ij} \leqslant l$ for all i, j, we have, by part (c) of Definition 4.3, that

$$|\{P_\alpha^{ij} : ij \notin X\}| < \varepsilon_1 \binom{t}{2} l,$$

which together with (4.3) concludes the proof of (i).

(ii) Let $\mathscr{J}$ be the set of all triads $P = (P_\alpha^{i,j}, P_\beta^{i,k}, P_\gamma^{j,k})$ such that $[i, j, k]^2 \cap ([t]^2 \setminus X) \neq \varnothing$ while all three members of P are ε_2-regular. Then, by the (δ, r)-regularity of $\mathscr{P}$, we infer that

$$\begin{aligned} \delta n^3 &> \sum \{t(P) : P \in \mathscr{I}\} \geqslant \sum \{t(P) : P \in \mathscr{I} \setminus \mathscr{J}\} \\ &\geqslant (1-2\varepsilon_2)(1/l - 2\varepsilon)^3\, m^3\, |\mathscr{I} \setminus \mathscr{J}|, \end{aligned}$$

where the last inequality follows by Proposition 4.2. Thus, we have

$$|\mathscr{I} \setminus \mathscr{J}| \leqslant \frac{\delta n^3}{(1-2\varepsilon_2)((1/l) - 2\varepsilon_2)^3\, m^3} \leqslant \frac{\delta t^3}{(1-2\varepsilon_2)((1/l)-2\varepsilon_2)^3} < 2\delta (lt)^3.$$

On the other hand, as $|[t]^2 \setminus X| \leqslant \varepsilon_1 \binom{t}{2}$, we conclude that

$$|\mathscr{J}| \leqslant l^3 \varepsilon_1 \binom{t}{2} (t-2) < \varepsilon_1 (lt)^3.$$

Hence,

$$|\mathscr{I}| \leqslant |\mathscr{I} \setminus \mathscr{J}| + |\mathscr{J}| < (2\delta + \varepsilon_1)\, t^3 l^3. \quad \blacksquare$$

As our main tool, we will use the following Hypergraph Regularity Lemma proved by Frankl and Rödl (cf. Theorem 3.12 in [6]).

THEOREM 4.7 [6]. *For all integers s, t_0, and l_0, for all δ and ε_1, $0 < \varepsilon_1 \leqslant \delta^4/s$, and for all integer valued functions $r = r(t, l)$ and all decreasing functions $\varepsilon_2(l)$ such that $0 < \varepsilon_2(l) \leqslant 1/l$, there exist T_0, L_0, and N_0 such that if $H_1, H_2, ..., H_s$ are 3-uniform hypergraphs on the same vertex set V with $|V| > N_0$, then, for some t, l satisfying $t_0 \leqslant t < T_0$, $l_0 \leqslant l < L_0$ there exists an equitable, $(l, t, \varepsilon_1, \varepsilon_2(l))$-partition which is $(\delta, r(t, l))$-regular with respect to each H_i, $i = 1, ..., s$.*

In the proof of Theorem 1.4 we shall also use another result from [6] (Lemma 4.2 there) which is an analog of Proposition 4.2.

Definition 4.8. Let V_1, V_2, V_3, V_4 be four disjoint sets of cardinality m. For each pair $1 \leqslant i < j \leqslant 4$, let P^{ij} be a bipartite graph with bipartition (V_i, V_j).

The six-tuple of bipartite graphs P^{ij} is called an (l, ε)-*sextet* if for all i, j, $1 \leqslant i < j \leqslant 4$, the graph P^{ij} is $(\varepsilon, (1/l))$-regular.

Consider a 4-partite 3-uniform hypergraph H with 4-partition $V_1 \cup V_2 \cup V_3 \cup V_4$, $|V_1| = |V_2| = |V_3| = |V_4| = m$. We say that an (l, ε)-sextet P^{ij}, $1 \leqslant i < j \leqslant 4$, and the hypergraph H form a $(\delta, r, \alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{234})$-*quartet of triads* if for every 3-element subset $\{i, j, k\} \subset \{1, 2, 3, 4\}$ the following two conditions hold:

(i) the triad $P^{ijk} = (P^{ij}, P^{ik}, P^{jk})$ is (δ, r)-regular,

(ii) $d_H(P^{ijk}) = \alpha_{ijk}$.

Lemma 4.9 [6]. *For all positive $\alpha_{123}, \alpha_{124}, \alpha_{134}$ and α_{234} there exists $\delta > 0$ ($\delta \leqslant (\alpha/10)^6$, where $\alpha = \min\{\alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{234}\}$) such that for every $l > 1/\delta$ there exist r ($r \leqslant \alpha^3 l^3/10$), and $c_0 > 0$ such that for every $\varepsilon \leqslant 1/l^{19}$ if an (l, ε)-sextet $P^{ij}, 1 \leqslant i < j \leqslant 4$, with vertex set $V_1 \cup V_2 \cup V_3 \cup V_4$, $|V_1| = |V_2| = |V_3| = |V_4| = m$, and a 4-partite 3-uniform hypergraph H form a $(\delta, r, \alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{234})$-quartet of triads then H contains at least $c_0 m^4$ copies of $K_4^{(3)}$.*

5. EDGE COLORING

In this section we give the proof of Theorem 1.4. But first we will convince the reader that $K_4^{(3)}$ is indeed the smallest nontrivial case.

To this end, consider the 3-uniform hypergraph obtained from $K_4^{(3)}$ by removing one edge. We shall call it, by analogy with the graph obtained from a triangle by removing one edge, *a hypercherry*. Let G be a hypercherry. Then, $m_G^3 = 2$ and thus, according to Conjecture 1.3 stated in Introduction and to the note thereafter, $n^{-1/2}$ should be the threshold for the property $K^{(3)}(n, p) \to (G)_r^e$.

To show that Conjecture 1.3 is true in this case, recall that $n^{-1/2}$ is the threshold for the graph Ramsey property $K(n, p) \to (K_3)_r^e$ (see [11]) and consider the pair neighborhood of a fixed vertex v defined as the set

$$N_v = \{\{u, w\} : uwv \in K^{(3)}(n, p)\}.$$

The set N_v is a random graph $K(n-1, p)$ and if $p > Cn^{-1/2}$ then, almost surely, for every r-coloring of its edges, and, in particular, for one which is naturally induced by an r-coloring of the triples of $K^{(3)}(n, p)$, there is a monochromatic triangle. This triangle corresponds to a monochromatic hypercherry.

For a hypercherry G, let us define its *missing triple* as the unique triple which together with G yields a copy of $K_4^{(3)}$. Hence, in case of $K_4^{(3)}$, one needs to show that for at least one monochromatic hypercherry, its missing triple also appears in $K^{(3)}(n, p)$ and, moreover, it is colored by the same color as the hypercherry.

Let us outline our proof first. Given a hypergraph K and a coloring $h: K \to \{1, 2\}$, we denote by $H(K, h)$ the set of the missing triples of all monochromatic hypercherries of K under h. It naturally splits into a (not necessarily disjoint) union $H_1 \cup H_2$ according to the color of the hypercherry.

In the proof of Theorem 1.4 we shall use the well known technique called *the two-round exposure*. Representing $p = p_1 + p_2 - p_1 p_2$, one first generates the random hypergraph $K^{(3)}(n, p_1)$, conditions on the outcome, colors it, and only then generates $K^{(3)}(n, p_2)$. We shall be assuming that both p_1 and p_2 are of the same order of magnitude as p, but that p_2 is sufficiently bigger than p_1.

We shall show that as a result of round 1, almost surely, for every 2-coloring, either H_1 or H_2 will contain $\Omega(n^4)$ copies of $K_4^{(3)}$ (say, this will be true for H_1). Then, in round 2, conditioning on the event that $K^{(3)}(n, p_1)$ satisfies the above property, and fixing a 2-coloring h, we apply Lemma 2.1 and conclude that with probability $1 - e^{-\Omega(n^3 p_2)}$ there is at least one copy, say K_0, of $K_4^{(3)}$ in the random hypergraph $(H_1)_{p_2}$.

The exponential probability of failure is necessary, as it must be multiplied by the number of possible colorings of $K^{(3)}(n, p_1)$, which is, almost surely, $2^{O(n^3 p_1)}$.

Finally, we complete that coloring. If at least one of the edges of K_0 is colored by color 1, it forms together with a hypercherry a copy of $K_4^{(3)}$ in color 1. Otherwise, K_0 is a hypercherry in color 2.

Formally, this outline can be described as follows. Let $\mathscr{A}$ be the event that $K^{(3)}(n, p) \nrightarrow (K_4^{(3)})_2^e$ and let $\mathscr{B}$ be the event that $|E(K^{(3)}(n, p_1))| < n^3 p_1$ and that for every $h: E(K^{(3)}(n, p_1)) \to \{1, 2\}$ there is an $s \in \{1, 2\}$ such that $H_s(K^{(3)}(n, p_1), h)$ contains at least cn^4 copies of $K_4{}^{(3)}$, for some constant $c > 0$. Conditioning on the outcome $K^{(3)}(n, p_1) = K$ of the first round, for every $h: E(K) \to \{1, 2\}$, let $\mathscr{A}_h$ be the event that there is an extension $\bar{h}: E(K^{(3)}(n, p)) \to \{1, 2\}$ of h such that $\bar{h} \equiv h$ on K and there is no monochromatic copy of $K_4^{(3)}$. Then

$$\mathrm{P}(\mathscr{A}) \leqslant \mathrm{P}(\neg \mathscr{B}) + \sum_{K \in \mathscr{B}} \mathrm{P}(\mathscr{A} \mid K)\, \mathrm{P}(K),$$

and

$$P(\mathscr{A} \mid K) = P\left(\bigcup_h \mathscr{A}_h \mid K\right) \leqslant 2^{n^3 p_1} P(\mathscr{A}_{h_0} \mid K),$$

where h_0 maximizes the conditional probability. Thus, all we have to show is that

(A) $P(\neg \mathscr{B}) = o(1)$

and that

(B) for every $K \in \mathscr{B}$ and for every 2-coloring h of the edges of K,

$$P(\mathscr{A}_h \mid K^{(3)}(n, p_1) = K) \leqslant 2^{-bn^3 p_2},$$

where b is an absolute constant.

As we mentioned before, (B) is an easy application of Lemma 2.1. Indeed, observe that, with X standing for the number of copies of $K_4^{(3)}$ in $(H_1)_{p_2}$, $E(X) \geqslant cn^4 p_2^4$ and the denominator of the exponent appearing in Lemma 2.1 can be bounded from above by $n^4 p_2^4 + n^5 p_2^7$. Thus, assuming that $p_2 > C_2 n^{-1/3}$, where $C_2 > 1$, we have

$$P((H_1)_{p_2} \not\supset K_4^{(3)}) = P(X = 0) < \exp\{-bn^3 p_2\},$$

with $b = \frac{1}{2}c^2$.

This determines the relationship between p_1 and p_2. As we need $2^{n^3 p_1} e^{-bn^3 p_2} = o(1)$, we impose, with some room to spare, that $p_2 > (2/c^2)\, p_1$. Hence, if statement (A) is true with a given c and with $p_1 > c_1 n^{-1/3}$, then our Theorem 1.4 is true with $C = c_1 + C_2 > c_1(1 + (2/c^2))$.

Please note that, as follows from a detailed analysis of the forthcoming proof, by making c_1 smaller we decrease c even more and, in effect, the constant C would grow.

The first component of the event $\mathscr{B}$, the inequality $|E(K^{(3)}(n, p_1))| < n^3 p_1$, is an immediate consequence of Chebyshev's inequality. The essential part of statement (A), saying that almost surely for every h: $E(K^{(3)}(n, p_1)) \to \{1, 2\}$ there is an $s \in \{1, 2\}$ such that $H_s(K^{(3)}(n, p_1), h)$ contains at least cn^4 copies of $K_4^{(3)}$, will follow from the next two claims.

Let us recall that for a graph G, $T(G)$ stands for the set of (the vertex sets of) its triangles. The notion of an (ε, d)-regular graph was defined in Section 4. We will also use the following related concept. A graph is said to be (ε, d, t)-*regular* if there is an equable partition $V_1, \ldots V_t$ of its vertex set such that each of the $\binom{t}{2}$ bipartite subgraphs spanned by the pairs of partition sets is (ε, d)-regular. Any partition which realizes this property is called *relevant*.

Let, for every $0<d<1$,

$$\varepsilon(d)=d^6/(10^6 S(10^{-5}d))$$

and $S(\varepsilon_0)$ is the Szemerédi constant appearing in the Szemerédi Regularity Lemma (Theorem 4.1).

We say that a hypergraph H has the property $\mathscr{P}(v, d_1, d_2)$ if for every $d \geqslant d_1$ and for every $(\varepsilon(d), d, 18)$-regular graph G with $V(G) \subseteq V(H)$ and $n/2 \geqslant |V(G)| \geqslant vn$, $|H \cap T(G)| > d_2 |T(G)|$.

Claim 5.1. For all v and d_1, almost surely, for every $h\colon K^{(3)}(n, p_1) \to \{1, 2\}$, where $p_1 > 10^{-100} n^{-1/3}$, the hypergraph $H(K^{(3)}(n, p_1), h)$ has property $\mathscr{P}(v, d_1, d_2)$, with $d_2 = 10^{-273}$.

Claim 5.2. For every d_2 there exist v, d_1, and $c>0$ such that every 2-coloring of a hypergraph with property $\mathscr{P}(v, d_1, d_2)$ results in at least cn^4 monochromatic copies of $K_4^{(3)}$.

It will follow from the proof that the constant c is very small, with d_2^4 being an obvious upper bound on it. Claims 5.1 and 5.2 together imply that there exists a constant c such that the random hypergraph $K^{(3)}(n, p_1)$ with $p_1 > 10^{-100} n^{-1/3}$, almost surely, has the property that for every 2-coloring of its edges, there is a color $s \in \{1, 2\}$ for which the hypergraph H_s build up from the missing triples of its monochromatic cherries in color s contains at least cn^4 copies of $K_4^{(3)}$. This is, however, the essential part of property $\mathscr{B}$ and so the statement (A) follows.

The rest of this section is devoted to proving these claims. Once they are proved, Theorem 1.4 is established.

Proof of Claim 5.1. We shall use the following result which will be proved in the Appendix.

THEOREM A.1. *For every $0<d<1$ there are constants m_0, b_0, and C_0 such that if G is an $(\varepsilon(d), d, 18)$-regular graph on $m > m_0$ vertices and $p > C_0 m^{-1/2}$, then, with probability at least $1-2^{-b_0 m^2 p}$, every 2-coloring of the random graph G_p results in at least $a(dmp/18)^3$ monochromatic triangles, where $a = 10^{-131}$.*

Let now v and d_1 be given. We are going to prove that for $p_1 > 10^{-100} n^{-1/3}$, almost surely, the following is true: for all $d \geqslant d_1$, for every $(\varepsilon(d), d, 18)$-regular graph G, with $vn < |V(G)| < n/2$, $V(G) \subset [n]$, and for every 2-coloring h of the edges of $K^{(3)}(n, p_1)$,

$$|H(K^{(3)}(n, p_1), h) \cap T(G)| > d_2 \, |T(G)| \tag{5.1}$$

where $d_2 = 10^{-273}$.

Let us fix an $(\varepsilon(d), d, 18)$-regular graph G, with $vn < |V(G)| < n/2$, $V(G) \subset [n]$, $d \geqslant d_1$. As there are less than 2^{n^2} graphs G with $V(G) \subset [n]$, we need to show that (5.1) holds with probability $1 - o(2^{-n^2})$.

Assume that G has a relevant partition $(V_1, ..., V_{18})$, where $|V_1| = \cdots = |V_{18}| = \bar{n}$.

If there is a hypercherry whose missing edge corresponds to a triangle of G and whose fourth vertex is not in $V(G)$ then we say that this triangle *supports* that hypercherry.

For every $v \in [n] \backslash V(G)$ define an independent copy of the random graph G_{p_1} by

$$G_{p_1}^v = \{uw \in E(G) \colon vuw \in K^{(3)}(n, p_1)\}.$$

Every triangle of $G_{p_1}^v$ supports a hypercherry with the fourth vertex v. Every coloring $h\colon K^{(3)}(n, p_1) \to \{1, 2\}$ imposes naturally a 2-coloring $h^v\colon G_{p_1}^v \to \{1, 2\}$.

By Theorem A.1, there exists a constant $b_0 = b_0(d)$ such that, with probability at least $1 - e^{-b_1 \binom{n}{3} p_1}$, $b_1 = 6\,\frac{1}{4} v^2 b_0$, for at least, say, $\frac{1}{2}(n - |V(G)|) > \frac{1}{4} n$ vertices $v \in [n] \backslash V(G)$, there are at least $a(d_1 \bar{n} p_1)^3$ monochromatic (under h^v) triangles, for every coloring $h\colon K^{(3)}(n, p_1) \to \{1, 2\}$, where $a = 10^{-131}$.

Each such triangle supports a monochromatic hypercherry in $K^{(3)}(n, p_1)$. However, one triangle may support many hypercherries. For every triangle T of G, let $x_T^{(1)}$ $(x_T^{(2)})$ count the hypercherries in color 1 (2) supported by T. Consequently, with probability at least

$$1 - e^{-b_1 \binom{n}{3} p_1}, \tag{5.2}$$

for every 2-coloring of $K^{(3)}(n, p_1)$,

$$\sum_{T \in T(G)} x_T \stackrel{\text{def}}{=} \max\left(\sum_{T \in T(G)} x_T^{(1)}, \sum_{T \in T(G)} x_T^{(2)}\right) > \frac{1}{8}\, na(d_1 \bar{n} p_1)^3. \tag{5.3}$$

We would like to show that

$$t \stackrel{\text{def}}{=} |\{T \colon x_T > 0\}| > d_2\, |T(G)|. \tag{5.4}$$

By Proposition 4.2,

$$|T(G)| < 2 \binom{18}{3} (d_1 \bar{n})^3. \tag{5.5}$$

Hence, by (5.3), (5.5) and our assumption on p_1, inequality (5.4) is certainly true if, say, $t > \frac{1}{3} \sum_{T \in T(G)} x_T$.

Otherwise, i.e when

$$t \leqslant \tfrac{1}{3} \sum_{T \in T(G)} x_T, \tag{5.6}$$

we need to show, that with probability high enough, (5.4) is still satisfied.

A double hypercherry is a pair of hypercherries supported by the same triangle of G. There are $\binom{x_T^{(i)}}{2}$ double hypercherries in color i supported by any given T, for $i = 1, 2$.

Let D denote the total number of double hypercherries. By (5.5), we have

$$\mathrm{E}(D) \leqslant \binom{n - \nu n}{2} |T(G)|\, p_1^6 < \binom{18}{3} n^2 \bar{n}^3 d_1^3\, p_1^6.$$

Assume for a moment that with probability close enough to 1, $D < 2\mathrm{E}(D)$. Then

$$\sum \binom{x_T}{2} < 2 \binom{18}{3} n^2 \bar{n}^3 (d_1)^3\, p_1^6 \tag{5.7}$$

and, by Jensen's inequality and by (5.3) and (5.6), we obtain that

$$\sum \binom{x_T}{2} > t \binom{\sum x_T/t}{2} > \frac{(\sum x_T)^2}{3t} > \frac{a^2 (d_1\, p_1)^6\, n^2 \bar{n}^6}{192t}. \tag{5.8}$$

This compared to (5.7) yields, by (5.5),

$$t > \frac{a^2}{384 \binom{18}{3}} (d_1 \bar{n})^3 > \frac{a^2}{768 \binom{18}{3}^2} |T(G)|, \tag{5.9}$$

which would prove Claim 5.1 with $d_2 = a^2/768\binom{18}{3}^2$.

Unfortunately, as mentioned in Section 2, we cannot claim the inequality $D < 2\mathrm{E}(D)$ with sufficiently high probability. Therefore, we need to refine our approach. For $E \subset [n]^3$, let D_E be the number of double hypercherries in $K^{(3)}(n, p_1) \backslash E$. We will show that, with probability at least $1 - e^{-\Omega(n^3 p_1)}$, there exists a set $E_0 \subset [n]^3$ such that $D_{E_0} < 2\mathrm{E}(D)$, while at the same time, for every 2-coloring of $K^{(3)}(n, p_1) \backslash E_0$, an inequality only slightly weaker than (5.3) is valid. This will enable us to literally repeat the argument leading to (5.8) and (5.9) with only minor adjustments.

To achieve that, we will apply Corollary 2.5 with $F = [n]^3$ and $\delta > 0$, so small that the inequality (2.1) holds with the constant b replaced by b_1 from (5.2). Furthermore, let $\mathscr{S}$ be the family of all double hypercherries supported by the triangles of G, and let property $\mathcal{Q}$ state that inequality (5.3) holds for every 2-coloring of the triples, with p_1 replaced by $p_0 = (1-\delta)\, p_1$. In other words, the property $\mathcal{Q}$ considered here is the family of all subsets $R \subseteq [n]^3$ such that for every partition $R = R_1 \cup R_2$ we have $\sum_{T \in T(G)} x_T^{(i)} > f(n) = (a/8)(d_1\, p_0(n))^3\, n\bar{n}^3$ in R_i for either $i=1$ or $i=2$. Note, that by fixing p_0, we made property $\mathcal{Q}$ independent from p and, therefore, it is increasing.

Switching from p_1 to p_0, (5.3) becomes,

$$\mathrm{P}(K^{(3)}(n, p_0) \in \neg\mathcal{Q}) < 2^{-b_1 \binom{n}{3}\, p_0}.$$

Now, by Corollary 2.5, with probability at least $1 - 2^{-b' \binom{n}{3}\, p_1} - 2^{-k/6} = 1 - o(2^{-n^2})$, there is a set of triples $E_0 \subseteq K^{(3)}(n, p_1)$, $|E_0| = k = \frac{1}{2}\delta\binom{n}{3}\, p_1^3$, such that both, (5.3) holds for $K^{(3)}(n, p_1) \backslash E_0$ with the extra factor of $(1-\delta)^3$ on the right, and $D_{E_0} < 2\mathrm{E}(D)$.

Thus, by Jensen's inequality, (5.8) and (5.9) hold, with the additional factor of $(1-\delta)$ raised to the appropriate power, and $d_2 = (a^2(1-\delta)^6/768\binom{18}{3}^2) > 2^{-273}$, for small enough δ. ∎

Proof of Claim 5.2. Set $x = 2R_3(18, 4, 4)$, where $R_3(18, 4, 4)$ is the Ramsey number assuring that coloring the triples from a set of size at least $R_3(18, 4, 4)$ by black, red and blue, always results in either an 18-element subset with all triples black or a 4-element subset with all triples red or a 4-element subset with all triples blue.

Furthermore, set $s=2$, $t_0 = x$, $l_0 = 1/\delta$, $\delta = \min\{(d_2/20)^6,\ 1/20x^2\}$, $\varepsilon_1 = \frac{1}{2}\delta^4$, $r(t, l) = r(l) = (d_2 l)^3/80$ and $\varepsilon_2(l) = (1/100 l^{19})\varepsilon(1/l)$, where the function $\varepsilon(d)$ was defined prior to the formulation of Claim 5.1.

Let T_0, L_0 and N_0 be the parameters resulting from Theorem 4.7. Set $v = 18/(T_0+1)$ and $d_1 = 1/L_0$ and consider a 3-uniform hypergraph H on at least N_0 vertices, holding property $\mathcal{P}(v, d_1, d_2)$, together with an arbitrary 2-coloring $H = H_1 \cup H_2$ of its edges. We apply Theorem 4.7 to H_1 and H_2, obtaining an equitable, $(l, t, \varepsilon_1, \varepsilon_2)$-partition $\mathcal{P} = \{P_\alpha^{ij}\colon\ 1 \leqslant i < j \leqslant t,\ 0 \leqslant \alpha \leqslant l_{ij} \leqslant l\}$, which is (δ, r)-regular with respect to both H_1 and H_2, where $x \leqslant t \leqslant T_0$ and $1/\delta \leqslant l \leqslant L_0$.

Let $V = V_0 \cup \cdots \cup V_t$ be the corresponding partition of vertices of H. Consider an auxiliary multigraph M with vertex set $[t] = \{1, 2, ..., t\}$ and with multiplicity at most l, in which every edge, say $\{i, j\}_\alpha$, corresponds to one of the graphs P_α^{ij}, $1 \leqslant \alpha \leqslant l_{ij}$. (Note that we ignore the set V_0 and all graphs P_0^{ij}, $1 \leqslant i < j \leqslant t$.)

Call an edge (a triangle) of M *bad* if it corresponds to an $(\varepsilon_2, 1/l)$-irregular graph ((δ, r)-irregular triad with respect to either H_1 or H_2) or *good* otherwise. Note that there is no direct relation between bad edges and bad triangles: 3 bad (good) edges may form a good (bad) triangle.

A subgraph of M is *clean* if all its edges and triangles are good. We shall find a clean, simple (i.e. with no parallel edges), complete subgraph $K_{x/2}$ of M.

To achieve this task we employ the probabilistic deletion method. We choose randomly x vertices of M and then we also choose one edge from between each pair of chosen vertices (if there is any), obtaining a random simple subgraph R of M.

The expected number of nonedges of R is, by Definition 4.3(c), not greater than

$$\varepsilon_1 \binom{t}{2} \frac{\binom{t-2}{x-2}}{\binom{t}{x}} < \frac{1}{2} \varepsilon_1 x^2.$$

For a given edge $e = \{i, j\}_\alpha \in M$, $\alpha \geqslant 1$,

$$\mathrm{P}(e \in R) = \frac{\binom{t-2}{x-2}}{\binom{t}{x}} \frac{1}{l_{ij}} = \frac{x(x-1)}{t(t-1)\, l_{ij}}.$$

Similarly, for a given triangle $T = \{\{i, j\}_\alpha, \{i, k\}_\beta, \{j, k\}_\gamma\}$ on vertices i, j, k,

$$\mathrm{P}(T \subset R) = \frac{\binom{t-3}{x-3}}{\binom{t}{x}} \frac{1}{l_{ij} l_{ik} l_{jk}} = \frac{x(x-1)(x-2)}{t(t-1)(t-2)\, l_{ij} l_{ik} l_{jk}}. \tag{5.10}$$

Now, due to the choice of $\varepsilon_2 = \varepsilon_2(l)$, we have both $\varepsilon_2 < 1/2l$ and

$$(1 - 2\varepsilon_2)(1 - 2\varepsilon_2 l)^3 > \tfrac{1}{2}$$

and hence, by Proposition 4.6, there are less than $3\varepsilon_1 \binom{t}{2} l$ bad edges and less than $2(2\delta + \varepsilon_1)\, t^3 l^3$ bad triangles in M.

Let Y be the set of all pairs $i, j \in [t]^2$ for which part (c) of Definition 4.3 holds, i.e., $|P_0^{ij}| \leqslant \varepsilon_1 m^2$, and for all $\alpha \geqslant 1$, $(1/l) - \varepsilon_2 \leqslant d_{P_\alpha^{ij}}(V_i, V_j) \leqslant (1/l) + \varepsilon_2$.

For $i, j \in Y$, we have $l_{ij}(1/l + \varepsilon_2) > 1 - \varepsilon_1$, and so

$$\frac{l}{l_{ij}} < \frac{1 + \varepsilon_2 l}{1 - \varepsilon_1} < \frac{4}{3}. \tag{5.11}$$

Thus the expected number of bad edges in R of the form $e=\{i,j\}_\alpha$, where $i,j\in Y$, is less than

$$\frac{x(x-1)}{t(t-1)\,l_{ij}}\,3\varepsilon_1\binom{t}{2}\,l<2\varepsilon_1 x^2.$$

For the remaining bad edges we have their expectation bounded by

$$\sum_{i,j\notin Y} l_{ij}\frac{x(x-1)}{t(t-1)\,l_{ij}}<\frac{1}{2}\,\varepsilon_1 x^2,$$

as, by part (c) of Definition 4.3, $|[t]^2\backslash Y|<\varepsilon_1\binom{t}{2}$.

As far as bad triangles are concerned, let $Z=\{\{i,j,k\}: ij, ik, jk\in Y\}$. Then,

$$|[t]^3\backslash Z|\leqslant\varepsilon_1\binom{t}{2}(t-2), \tag{5.12}$$

and the expected number of bad triangles $T=\{\{i,j\}_\alpha,\ \{i,k\}_\beta,\ \{j,k\}_\gamma\}$, with $\{i,j,k\}\in Z$, is bounded, due to (5.11), by

$$\frac{x(x-1)(x-2)}{t(t-1)(t-2)\,l_{ij}l_{ik}l_{jk}}\,2(2\delta+\varepsilon_1)\,t^3l^3<(4/3)^3\,2(2\delta+\varepsilon_1)\,x^3.$$

The expected number of other bad triangles is, by (5.10) and (5.12), not greater than

$$\sum_{i,j,k\notin Z} l_{ij}l_{ik}l_{jk}\frac{x(x-1)(x-2)}{t(t-1)(t-2)\,l_{ij}l_{ik}l_{jk}}<\frac{1}{2}\,\varepsilon_1 x^3.$$

Altogether, the total expected number of non-edges, bad edges, and bad triangles in R is at most

$$3\varepsilon_1 x^2+(2(4/3)^3+1/2)\,\varepsilon_1 x^3+4(4/3)^3\,\delta x^3\leqslant\frac{x}{2}$$

by our choice of δ and ε_1.

By deleting at most $x/2$ vertices from R we obtain a clean $(x/2)$-clique $K_{x/2}$.

Assume without loss of generality that $V(K_{x/2})=\{1,2,...,x/2\}$. Let P^{ij}, $1\leqslant i<j\leqslant x/2$, be the bipartite graphs corresponding to the edges of $K_{x/2}$. Consider now an auxiliary coloring of $[V(K_{x/2})]^3$ by red, blue, and black. We color ijk red (blue) if

$$|H_s\cap T(P^{ij},P^{ik},P^{jk})|>\frac{d_2}{2}\,|T(P^{ij},P^{ik},P^{jk})|,$$

$s=1$ or 2, respectively, and black otherwise. Since $n/36 > m = |V_i| \geqslant (n-t+1)/t > \nu n/18$, and since H satisfies property $\mathscr{P}(\nu, d_1, d_2)$, it is impossible that there are 18 indices $i_1, ..., i_{18}$ such that all $\binom{18}{3}$ triples they form are colored black. Indeed, this would mean that there is an $(\varepsilon(d), d, 18)$-regular graph, where $d=1/l \geqslant d_1$, with only at most a d_2-fraction of its triangles captured by H—a contradiction with the property $\mathscr{P}(\nu, d_1, d_2)$. (Note that $\varepsilon_2(l) < \varepsilon(1/l)$.)

Hence, by the definition of x, there are 4 indices i, j, k, h such that all four triples ijk, ijh, ikh, jkh are colored by the same color, red or blue. Without loss of generality let this color be red. Let us relabel the indices i, j, k, h as 1,2,3,4 and look closely at the obtained configuration: There are 4 sets V_1, V_2, V_3, V_4 and 6 bipartite $(\varepsilon_2(l), 1/l)$-regular graphs P^{ij}, $1 \leqslant i < j \leqslant 4$. In other words, they form an $(l, \varepsilon_2(l))$-sextet.

Moreover, all four triads $P_{ijk} = (P^{ij}, P^{ik}, P^{jk})$, $i, j, k \in \{1, 2, 3, 4\}$, are (δ, r)-regular with respect to H_1, and with density $d_{H_1}(P_{ijk}) > d_2/2$, where $\delta \leqslant (d_2/20)^6$ and $r(l) = d_2^3 l^3/80$. By Lemma 4.9 we finally conclude that there are at least $c_0 m^4 = cn^4$ copies of $K_4^{(3)}$ in H_1. This proves Claim 5.2 and completes the proof of Theorem 1.4. ∎

APPENDIX

In the proof of Claim 5.1 we utilized a refinement of our result from [11] about monochromatic triangles in colorings of random graphs. For a given graph G, the random graph G_p is obtained by independent deletion of each edge from G with probability $1-p$. Recall that $\varepsilon(d) = d^6/(10^6 S(10^{-5}d))$, where the Szemerédi constant $S(\varepsilon_0)$ was defined through Theorem 4.1.

THEOREM A.1. *For every* $0 < d < 1$ *there are constants* n_0, b, *and* C *such that if* G *is an* $(\varepsilon(d), d, 18)$-*regular graph on* $n > n_0$ *vertices and* $p > Cn^{-1/2}$, *then, with probability at least* $1 - 2^{-bn^2p}$, *every* 2-*coloring of the random graph* G_p *results in at least* $a(dnp/18)^3$ *monochromatic triangles, where* $a = 10^{-131}$.

Note that, by Proposition 4.2, G contains approximately $\binom{18}{3}(dn/18)^3$ triangles and thus the theorem asserts that, in the random subgraph G_p, the number of monochromatic triangles created by an arbitrary coloring will be at least a fraction of the expected number of all triangles in G_p. We are not aiming at the best possible constant a; for our purpose it is enough if we just know that a does not depend on anything.

The same result is true for arbitrary number of colors, but we do not need it here. The proof below is to some extent based on Joel Spencer's version of our proof from [11] (see [13]). It is, in principle, parallel to the proof of Theorem 1.4 presented in Section 5.

There is no magic in number 18. The reason we picked it is that $18 = 1 + R(3, 3, 3)$. Theorem A.1 is, of course, true for every $(\varepsilon(d), d, f)$-regular graph G with $f \geqslant 18$ and $a = a_f$.

Our earlier result from [11] was not strong enough, since it only claimed the presence of at least one monochromatic triangle in each coloring. Neither was our general result from [12], since the constant a there, after dividing by d^3, still depended in an uncontrolled way on d. The only previous result which provided an independent constant a was that in [5, 9], but there the probability of failure was not exponentially small with respect to the number of edges of the random graph. Hence, we are destined to prove Theorem A.1 here.

Proof. Let G be an $(\varepsilon(d), d, 18)$-regular graph with vertex set V and let $(V_0, ..., V_{17})$ be a relevant partition of G, $|V_0| = \cdots = |V_{17}| = n/18 = \bar{n}$. We assume that the sets V_i, $i = 0, ..., 17$ are independent sets of G.

We employ a variant of the two-round exposure, with round 1 taking care of the edges between V_0 and $\bigcup_{i=1}^{17} V_i$ only. Let us denote this bipartite subgraph of G by G^0.

We expose the edges of G^0 with probability $p_1 = \alpha p$ which is a suitable fraction of p to be determined later.

A cherry is a pair of edges sharing one endpoint which belongs to V_0. We say that an edge *supports* a cherry if together with that cherry it forms a triangle.

For a 2-coloring of the edges of a random graph $G^0_{p_1}$, we call an edge of G *i-friendly* if it supports at least $(1/50)\, d^2 \bar{n} p_1^2$ cherries in color i, $i = 1, 2$.

Let G^i be the subgraph of G consisting of the i-friendly edges, $i = 1, 2$.

Let $\mathscr{A}$ be the event there exists a 2-coloring of the edges of G_p which results in less than $a(d\bar{n}p)^3$ monochromatic triangles. Let $\mathscr{B}$ be the event that $e(G^0_{p_1}) < 20\bar{n}^2\, dp_1$ and that for every 2-coloring of the edges of $G^0_{p_1}$ either G^1 or G^2 contains at least $\frac{1}{5}(10^{-4}\bar{n}d)^3$ triangles.

Conditioning on the outcome $K = G^0_{p_1}$, for every $h: E(K) \to \{1, 2\}$, let $\mathscr{A}_h$ be the event that there is an extension $\bar{h}: E(G_p) \to \{1, 2\}$ of h such that there are less than $a(d\bar{n}p)^3$ monochromatic triangles.

Then

$$P(\mathscr{A}) \leqslant P(\neg \mathscr{B}) + \sum_{K \in \mathscr{B}} P(\mathscr{A} \mid K)\, P(K),$$

and, for $K \in \mathscr{B}$,

$$P(\mathscr{A} \mid K) = P\left(\bigcup_h \mathscr{A}_h \mid K\right) \leqslant 2^{20\bar{n}^2\, dp_1} P(\mathscr{A}_{h_0} \mid K),$$

where h_0 maximizes the conditional probability. Thus, all we have to show is that

(A) $P(\neg \mathscr{B}) < 2^{-b_1 n^2 p_1}$, and that

(B) for every $K \in \mathscr{B}$ and for every 2-coloring h of the edges of K, $P(\mathscr{A}_h \mid G^0_{p_1} = K_1) \leqslant 2^{-b_2 n^2 p}$, where b_2 is a constant.

We begin with proving (A), which will keep us busy for a while.

For a given integer s, a *sausage* is a 17-tuple of sets $U_1, ..., U_{17}$, with $U_i \subset V_i$ and $|U_i| = \bar{n}/s$ for all $i = 1, ..., 17$.

For a sausage $(U_1, ..., U_{17})$ and a 2-coloring of $G^0_{p_1}$, we call a triple $1 \leqslant i < j < k \leqslant 17$ *friendly* if at least one of the induced bipartite graphs $G[U_i, U_j]$, $G[U_i, U_k]$, or $G[U_j, U_k]$, contains at least $(d/10^4)(\bar{n}/s)^2$ r-friendly edges for $r = 1$ or $r = 2$.

LEMMA A.2. *For every $0 < d < 1$, and for every integer $s \leqslant S(10^{-5}d)$, there are constants b, C_1, and n_0 such that for every $(\varepsilon(d), d, 18)$-regular graph G on $n > n_0$ vertices with relevant partition $(V_0, ..., V_{17})$, and for every $p_1 > C_1 n^{-1/2}$, with probability at least $1 - 2^{-bn^2 p_1}$, the following is true. For every choice of a sausage and for each 2-coloring of the edges of $G^0_{p_1}$ all $\binom{17}{3}$ triples are friendly.*

Proof of Lemma A.2. Given d and s consider an $(\varepsilon(d), d, 18)$-regular graph G on $n > n_0$ vertices. Throughout we shall be abbreviating $\varepsilon = \varepsilon(d)$. Fix one of at most 2^n possible sausages $\mathscr{U} = (U_1, ..., U_{17})$.

As $G[V_0, U_i]$ is a subgraph of an (ε, d)-regular graph $G[V_0, V_i]$, at least $(1 - 17\varepsilon)\bar{n}$ vertices of V_0 each have at least $m = (d - \varepsilon)(\bar{n}/s)$ neighbors in each set U_i, $i = 1, ..., 17$. Let us denote the set of those vertices by $\bar{V}_0$ and for each $v \in \bar{V}_0$ let $N_v(i)$ be a set of m of its neighbors in U_i, while for each $u \in \bigcup_{j=1}^{17} U_j$, $N_u(i)$ denotes the set of *all* neighbors of u in U_i.

Observe that for every $v \in \bar{V}_0$ and for each pair k, l, the bipartite subgraph of G spanned by the sets $N_v(k)$ and $N_v(l)$, $G[N_v(k), N_v(l)]$, is $(\varepsilon s/(d - \varepsilon), \varepsilon, d)$-regular.

DEFINITION A.3. The set $W_v(k, l)$ consists of "wrong" pairs, i.e. of all pairs $\{u, w\} \in N_v(k)$ such that either

$$|N_v(l) \cap N_u(l)| > (d + \varepsilon) m$$

or

$$|N_v(l) \cap N_w(l)| > (d + \varepsilon) m$$

or

$$|N_v(l) \cap N_u(l) \cap N_w(l)| < (d - \varepsilon)^2 m.$$

By Lemma 2.1b of [11], the set $W_v(k, l)$ has cardinality smaller than $6(\varepsilon s/(d-\varepsilon))\binom{m}{2} < \eta m^2$, where $\eta = 3(\varepsilon s/(d-\varepsilon))$.

In our analysis we may actually focus only on the edges of G which connect the vertices of $\bar{V}_0$ with the corresponding sets $N_v(i)$, $i = 1, ..., 17$. Let us denote the graph of these edges by $G^0[\mathscr{U}]$.

For fixed k and l let X and Y be random subsets of $N_v(k)$ and $N_v(l)$, resp., resulting from round 1, i.e., X and Y are the sets of neighbors of vertex v in the random graph $G^0[\mathscr{U}]_{p_1}$ which belong to U_k and U_l, respectively.

We are aiming to show some regularity of $G[X, Y]$ by means of the following Lemma proved (as Proposition 2.6) in [3], which can be also deduced from Lemma 3.2 of [1].

Lemma A.4. *If* $G = (X, Y, E)$ *is a bipartite graph with at least* $(1 - 5\gamma_0)\,|X|^2/2$ *pairs* $\{u, w\}$ *of vertices of* X *satisfying* $\deg(u), \deg(w) > (d - \gamma_0)\,|Y|$ *and* $\deg(u, v) < (d + \gamma_0)^2\,|Y|$, *then* G *is* $((16\gamma_0)^{1/5}, d)$*-regular.*

Let $\gamma_2 = \varepsilon/(d - 2\varepsilon)$, and $\gamma_1 = \eta/(1 - \gamma_2)^2$, where $\eta = 3(\varepsilon s/(d - \varepsilon))$ was defined after Definition A.3.

Definition A.5. Let $Z_u = |Y \cap N_u(l)|$ and $Z_{u,w} = |Y \cap N_u(l) \cap N_w(l)|$. We denote by $RW_v(k, l)$ the random set of "wrong pairs", i.e. all pairs $\{u, w\} \in [X]^2$ such that either

$$Z_u < (d - \gamma_1)\,|Y|$$

or

$$Z_w < (d - \gamma_1)\,|Y|$$

or

$$Z_{u,w} > (d + \gamma_1)^2\,|Y|.$$

Note that in Definition A.5 pairs are wrong exactly in the opposite sense to those in the set $W_v(k, l)$ defined in Definition A.3. Later we will take advantage of this fact.

Definition A.6. A vertex $v \in \bar{V}_0$ is said to be *good* if for all $1 \leqslant k < l \leqslant 17$,

(i) $|X - mp| < \gamma_2 mp$, and

(ii) $|RW_v(k, l)| < 9\gamma_1\,|X|^2$.

Fact A.7. There is a constant b_3 such that for every $v \in \bar{V}_0$,

$$\mathrm{P}(v \text{ is not good}) < e^{-b_3 np}.$$

Proof. By Chernoff's inequality, $\mathrm{P}(\neg(i)) < 17 e^{-Ch(\gamma_2)\, mp}$.
As

$$\mathrm{P}(\neg(i) \cup \neg(ii)) = \mathrm{P}(\neg(i)) + \mathrm{P}(\neg(ii) \cap (i)),$$

we need an exponential upper bound on $\mathrm{P}(\neg(ii) \cap (i))$.

Let $\mathscr{E}_1$ be the event that $|[X]^2 \cap W_v(k, l)| < 9\gamma_1 |X|^2$, and let $\mathscr{E}_2$ be the event $([X]^2 \setminus W_v(k, l)) \cap RW_v(k, l) = \varnothing$.

Clearly, $\mathscr{E}_1 \cap \mathscr{E}_2$ implies (ii) and we need exponential estimates on $\mathrm{P}(\neg\mathscr{E}_1 \cap (i))$ and $\mathrm{P}(\neg\mathscr{E}_2 \cap (i))$.

We have

$$\mathrm{P}(\neg\mathscr{E}_1 \cap (i)) \leqslant \sum_{|t - mp| < \gamma_2 mp} \mathrm{P}(\neg\mathscr{E}_1 \mid |X| = t)\, \mathrm{P}(|X| = t).$$

In order to estimate $\mathrm{P}(\neg\mathscr{E}_1 \mid |X| = t)$ we shall now recall a result from [11, Proposition 1, page 267].

Proposition A.8. *Let, for a graph G, $e_G < \eta v_G^2$, and let R be a random subset of $V(G)$ of size $|R| = t$. There exists an absolute constant b_4 such that*

$$\mathrm{P}(e_{G[R]} > 3\eta t^2) < e^{-b_4 t}.$$

Applying Proposition A.8 with G being $W_v(k, l)$ and R being X conditioned on $|X| = t$, we immediately conclude that

$$\mathrm{P}(\neg\mathscr{E}_1 \cap (i)) < \exp\{-b_4(1 - \gamma_2)\, mp\}.$$

Turning to the event $\mathscr{E}_2$, observe that, while the random set X and the event (i) depend on the edges connecting v and $N_v(k)$ only, the event $\mathscr{E}_2$, conditioned on the choice of X, depends exclusively on the edges connecting v to $N_v(l)$. We thus have

$$\mathrm{P}(\neg\mathscr{E}_2 \cap (i)) = \sum_{|t - mp| < \gamma_2 mp} \sum_{X \in [N_v(k)]^t} \mathrm{P}(\neg\mathscr{E}_2 \mid X)\, \mathrm{P}(X)$$

and

$$\mathrm{P}(\neg\mathscr{E}_2 \mid X) \leqslant |X|^2\, \mathrm{P}(u, w \in RW_v(k, l)),$$

for a fixed pair $\{u, w\} \in [X]^2 \setminus W_v(k, l)$.

By Definition A.5, both Z_u and Z_w are binomially distributed with expectation at least $(d-\varepsilon)\,mp$, while $Z_{u,w}$ is binomially distributed with expectation at most $(d+\varepsilon)^2\,mp$. Hence, again by Chernoff's inequality, with probability at least $1-e^{-b_5 mp}$, $Z_u, Z_w > (1-\gamma_2)(d-\varepsilon)\,mp$, $Z_{u,w} < (1+\gamma_2)(d+\varepsilon)^2\,mp$, and also $(1-\gamma_2)\,mp < |Y| < (1+\gamma_2)\,mp$.

These inequalities imply that $\{u, w\} \notin RW_v(k, l)$, as γ_1 and γ_2 were chosen so that

$$(1-\gamma_2)(d-\varepsilon) > (d-\gamma_1)(1+\gamma_2) \tag{A.1}$$

and

$$(1+\gamma_2)(d+\varepsilon)^2 < (1-\gamma_2)(d+\gamma_1)^2 \tag{A.2}$$

hold. An easy way to verify (A.1) and (A.2) is to bound γ_1 by 3ε from below and solve both inequalities for γ_2. Out of two upper bounds we obtain this way, the one corresponding to (A.1) supersedes the other one and coincides with our choice of γ_2. ∎

Observe that if $v \in V_0$ is a good vertex, then, by Lemma A.4, the subgraph $G[X, Y]$ is, for all k, l, (ε_1, d)-regular with $\varepsilon_1 = (16\gamma_0)^{1/5}$, where $\gamma_0 = (18/5)\,\gamma_1$.

Now, assuming that $17\varepsilon < 10^{-5}$, we derive that, with probability $1-e^{-b_6 n^2 p}$, there are at least $0.9999\bar{n}$ good vertices in V_0.

For each good vertex v and for every triple $1 \leqslant i < j < k \leqslant 17$, let us determine the majority color on the edges between v and each of U_t, $t = i, j, k$. (In the case of a tie we choose a color arbitrarily.)

Without loss of generality we may assume that, for at least $\frac{1}{6}(0.9999\bar{n})$ good vertices, the first color dominates between v and both U_i and U_j.

As v is good, the subgraph $G[X, Y]$ is (ε_1, d)-regular and so at least $\frac{1}{4}(d-\varepsilon_1)(1-\gamma_2)^2\,(mp)^2$ edges of $G[U_i, U_j]$ support cherries of color 1 hanging at v.

Before engaging into tedious calculations, recall that $\varepsilon = d^6/(10^6 S(10^{-5}d)) \ll d$ and therefore also $\varepsilon_1 \leqslant d10^{-5} \ll d$. Moreover, $\gamma_2 = \varepsilon/(d-2\varepsilon) \ll 1$. Hence, by increasing some coefficients just a little, we may suppress ε, ε_1 and γ_2 in what follows.

For $e \in G[U_i, U_j]$, let x_e be the number of cherries in color 1 supported by edge e. So far we know that, with probability at least

$$1-e^{-b_6 n^2 p},$$

$$\sum_e x_e > \frac{0.9999}{4(6)}\,(d-\varepsilon_1)(d-\varepsilon)^2\,(1-\gamma_2)^2\,(\bar{n})^3\,(p_1/s)^2 > \frac{1}{25}\,d^3\bar{n}^3(p_1/s)^2. \tag{A.3}$$

A double cherry supported by a pair e is a 4-cycle containing e as a non-edge. Let D count the double cherries of $G^0[\mathcal{U}]_{p_1}$ supported by the edges of $G[U_i, U_j]$. Then

$$\mathrm{E}(D) \leqslant \binom{\bar{n}}{2} (d+\varepsilon)((d+\varepsilon)^2 \bar{n}/s)^2 p_1^4.$$

Chances are that D does not exceed its expectation too much. Assume for a moment that $D < 2\mathrm{E}(D)$ with probability high enough. Then

$$\sum_e \binom{x_e}{2} < 2\mathrm{E}(D) \leqslant \bar{n}^2(d+\varepsilon)((d+\varepsilon)^2 \bar{n}/s)^2 p_1^4 < 1.0001\bar{n}^4 d^5 p_1^4/s^2. \tag{A.4}$$

As the number L of edges of the graph $G[U_i, U_j]$ is less than $(d+\varepsilon)(\bar{n}/s)^2 < 1.0001 d(\bar{n}/s)^2$, inequality (A.3) implies that an average x_e is, roughly, about $(d^2/25)\,\bar{n}p_1^2$. Our ultimate goal is to show (with the help of (A.4)) that at least $10^{-4} d(\bar{n}/s)^2$ x_e's are larger than half of the average.

Let us order the x_e's from high to low

$$x_1 \geqslant \cdots \geqslant x_l \geqslant \frac{d^2}{50}\,\bar{n}p^2 > x_{l+1} \geqslant \cdots \geqslant x_L, \tag{A.5}$$

hoping to prove that the index l defined by (A.5) is at least a 10^{-4} fraction of L.

We have

$$\begin{aligned}\sum_{e=1}^{l} x_e &\geqslant \frac{1}{28}\,d^3\bar{n}^3(p_1/s)^2 - \frac{1}{56}\,(L-l)\,d^2\bar{n}p_1^2 \geqslant \frac{1}{25}\,d^2\bar{n}p_1^2\left(\frac{d^2\bar{n}^2}{s^2} - \frac{l}{2}\right)\\ &\geqslant \frac{1}{51}\,d^3\bar{n}^3(p_1/s)^2 > \frac{C_1^2}{18(51)}\,d^3(\bar{n}/s)^2,\end{aligned}$$

where we recall that $p_1 > C_1 n^{-1/2}$. If $l > \frac{1}{3}\sum_{e=1}^{l} x_e$ then $l > 10^{-4}L$, provided $C_1^2 > 3(18)(52)/10^4 d^3$.

Otherwise, by Jensen's inequality

$$\sum_{e=1}^{l} \binom{x_e}{2} \geqslant \frac{1}{3l}\left(\sum_{e=1}^{l} x_e\right)^2 \geqslant \frac{1}{3l}\left[\frac{1}{51}\,d^3\bar{n}^3(p_1/s)^2\right]^2.$$

which compared with (A.4) forces $l > (d/3(51)^2)(\bar{n}/s)^2 > 10^{-4}L$.

But for inequality $D < 2\mathrm{E}(D)$ to hold with probability exponentially close to 1, one needs to delete a few edges from $G^0[\mathcal{U}]_{p_1}$, i.e. apply Corollary 2.5.

We apply it to F being the set of edges of $G^0[\mathscr{U}]$, $\mathscr{S}$—the family of all 4-cycles there, and property $\mathscr{Q}$ stating that for every 2-coloring, the inequality (A.1) holds with p_1 replaced by $p_0 = (1-\delta)\, p_1$.

We have just proved that

$$\mathrm{P}(G^0[\mathscr{U}]_{p_0} \in \neg\mathscr{Q}) < 2^{-b_6 n^2 p_0} = 2^{-b_7 e(G^0[\mathscr{U}])\, p_0}.$$

Now, let δ satisfy inequality (2.1) with $b = b_7$. Then, by Corollary 2.5, with probability at least $1 - 2^{-b_1 n^2 p_1}$, $b_1 = b_1(\delta, d, s, b_7)$, both, (A.3) (with p_1 replaced by p_0) and (A.4) hold for $G^0[\mathscr{U}]_{p_1} \setminus E_0$ for some E_0 with $|E_0| = \frac{1}{2}\delta\, |V(G^0[\mathscr{U}])|\, p_1$, and our previous conclusion that $l > 10^{-4} d(\bar{n}/s)^2$ stands true for sufficiently small δ. ∎

Having proved Lemma A.2, we may resume the proof of Theorem A.1, still in the part devoted to showing statement (A).

Recall that G^r is the subgraph of $G[\bigcup_{i=1}^{17} V_i]$ consisting of r-friendly edges, $r = 1, 2$.

Now apply Szemerédi's Regularity Lemma (Theorem 4.1) to the pair (G^1, G^2) with $t = 17$ and $\varepsilon_0 = 10^{-5} d$, obtaining an equible partition of each V_i, $i = 1, ..., 17$, into s sets $U_i^1, ..., U_i^s$, $1 \leqslant s \leqslant S(\varepsilon_0)$, such that all but $\varepsilon_0 \binom{17}{2} s^2$ pairs U_i^k, U_j^l span ε_0-regular subgraphs of both G^1 and G^2 of, a priori, unknown density. (Please note that ε_0 is much bigger than the original ε, but still 10 times smaller than the density of at least one graph in a friendly triple.)

There are altogether s^{17} sausages. We consider a sausage $(U_1^{k_1}, ..., U_{17}^{k_{17}})$ to be *spoiled* if for some i, j the pair $U_i^{k_i}$, $U_j^{k_j}$ is not ε_0-regular either in G^1 or G^2. At most $\varepsilon_0 \binom{17}{2} s^{17}$ sausages are spoiled. Fix one sausage which is not spoiled, $U_1, ..., U_{17}$, say.

Now we shall use the whole power of number 17. For each $\{i, j\} \in [17]^2$, color it red (blue) if at least $10^{-4} d(\bar{n}/s)^2$ edges of $G[U_i, U_j]$ belong to G^1 (G^2). Color it black otherwise. Since $R(3, 3, 3) = 17$, there are 3 indices i, j, k such that all 3 pairs ij, ik, jk are colored the same.

By Lemma A.2 we know that, with high probability, this common color cannot be black, as each triple $1 \leqslant i < j < k \leqslant 17$ is friendly. Then it must be red, say.

Hence, we sorted out 3 bipartite subgraphs of G^1 (spanned by (U_i, U_j), (U_i, U_k), and (U_j, U_k)) which are ε_0-regular with density at least $10^{-4} d$. By Proposition 4.2 they span at least $\frac{1}{2}(10^{-4} d\bar{n}/s)^3$ triangles in color 1.

What we just proved for one sausage remains true for at least half of all $(1 - \varepsilon_0 \binom{17}{2})\, s^{17}$ unspoiled sausages, since we may assume that the first color dominates.

As each triangle may belong to at most s^{14} different sausages, we conclude that, with required probability, for every 2-coloring there are

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